the meniscus and the use of R_m as a characteristic parameter of the meniscus loses its meaning.

LITERATURE CITED

1. Chun Huh and L. E. Scriven, "Hydrodynamic model of steady movement of a solid/liquid/fluid contact line," J. Colloid Interface Sci., 35, No. 1 (1971).
2. B. V. Zheleznyi, "Motion of an arriving meniscus of the liquid in a capillary taking account of the specific properties of thin films," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1976).
3. B. V. Zheleznyi, "The hydrodynamics of the arriving meniscus of a wetting liquid," Dokl. Akad. Nauk SSSR, 219, No. 1 (1974).
4. R. I. Hansen and T. Y. Toong, "Dynamic contact angle and its relationship to forces of hydrodynamic origin," J. Colloid Interface Sci. 37, No. 1 (1971).
5. B. V. Zheleznyi, "Investigation of wetting films by the capillary method," in: Surface Phenomena in Liquids [in Russian], No. 3, Izd. Leningr. Univ. Leningrad (1975).

FINITE-AMPLITUDE INTERNAL WAVES AT AN INTERFACE BETWEEN TWO HEAVY LIQUIDS

S. I. Plaksin

UDC 532.591

The problem of steady-state waves at an interface between two heavy liquids has been discussed in several papers (see, e.g., [1, 2]). Here a method is proposed on the basis of reduction of the problem to the solution of a nonlinear conjugation problem.

Let us consider the flow of two incompressible liquids of different densities in a gravity field with specified velocities at an infinite distance from the interface. We consider the motion to be irrotational and assume that the interface line l, which moves at a certain horizontal velocity U without changing shape, is a Lyapunov curve with period λ. We set up a coordinate system OXY moving in the direction of wave propagation with velocity U. We assume that the absolute particle velocity of the liquid at the interface differs from the wave-propagation velocity. Under this condition the waves are nonbreaking [3].

We place the origin at the average level of the liquid interface line, directing the axis OX along the horizontal in the direction of absolute motion of the line l, and the axis OY along the vertical upward through one of the wave crests (Fig. 1). By Ω_k, k = 1, 2, we denote the domains with period λ occupied by the upper and lower liquids. We introduce the complex variables $Z_k = X_k + iY$ in Ω_k, corresponding to the complex-valued potentials $W_k = \Phi_k + i\Psi_k$ and complex velocities $V_k = dW_k/dZ_k$. We denote the absolute velocities of the liquids at an infinite distance from the interface by $V_{k\infty}$ and the densities by $\rho_k (\rho_1 < \rho_2)$.

We transform to dimensionless variables, putting $V_k = v_k V_{1\infty}$, $Z_k = z_k\lambda/2\pi$, and $W_k = w_k V_{1\infty}\lambda/2\pi$.

Under the stated assumptions the problem reduces to the determination of the wave profile and functions v_k that are analytic in Ω_k and satisfy the kinematic and dynamic conditions at l as well as the following condition at an infinite distance from the interface:

$$\begin{gathered}\psi_1 = \psi_2 = 0 \text{ at } l;\\ \operatorname{Im}(z) = [m_1|v_1(z)|^2 - (1+m_1)|v_2(z)|^2]\,\mathrm{Fr}/2\gamma^2 + c,\ z\in l;\\ v_1 \to 1-\gamma,\ y_1\to\infty;\ v_2\to\delta-\gamma,\ y_2\to-\infty,\end{gathered} \tag{1}$$

where $\mathrm{Fr} = U^2 2\pi/g\lambda$; $m_1 = \rho_1/(\rho_2 - \rho_1)$; $\gamma = U/V_{1\infty}$; $\delta = V_{2\infty}/V_{1\infty}$; g is the acceleration of gravity; and c is a certain functional.

We investigate the auxiliary plane of the complex variable u. Let the domain D^+ be the interior of the unit disk with center at the point u = 0 and D^- the exterior of the disk with

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 123-129, July-August, 1978. Original article submitted June 14, 1977.

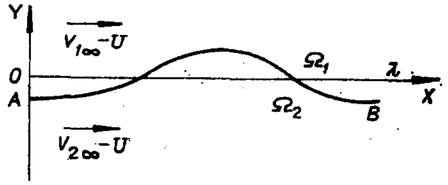

Fig. 1

cuts from zero to one and from one to infinity, respectively. We map the domain D^+ (D^-) onto $\Omega_1(\Omega_2)$ in such a way that the points A and B (Fig. 1) will correspond to points e^{i0}, $e^{i2\pi}$, an infinitely distant of Ω_1 will correspond to the point $u = 0$, and an infinitely distant point of Ω_2 will correspond to an infinitely distant point in the plane of u. The required mapping $f_1(u)$ $[f_2(u)]$ has the form [4]

$$f_1(u) = -i(\ln u + \omega_1(u)) \; (f_2(u) = -i(\ln u + \omega_2(u))),$$

where ω_1 is a function regular inside the disk $|u| < 1$ (ω_2 is a function regular outside $|u| \leqslant 1$). Here the wave profile goes over to a circle L of unit radius. Invoking the Kellogg theorem [5, 6] and the smoothness of the line l, we can show that the functions $df_k/d\tau$ satisfy at L the Hölder condition with exponent $\alpha (0 < \alpha \leqslant 1)$, $df_k/d\tau \neq 0$ at L, df_1/du is continuous for $u \neq 0$ in the disk $|u| \leqslant 1$, df_2/du is continuous outside $|u| < 1$, and the following relation holds:

$$\lim_{u \to \tau} df_k/du = df_k/d\tau \tag{2}$$

(here and elsewhere $d/d\tau$ is interpreted as the derivatives of limiting values, $\tau = e^{i\sigma}$, $\sigma \in [0, 2\pi]$).

We introduce the shift function $\beta(t) = \tau$ ($t = e^{is}$, $s \in [0, 2\pi]$):

$$\beta(t) = f_2^{-1}(f_1(t)). \tag{3}$$

Differentiating relation (3), we obtain

$$\beta'(t) df_2(\beta(t))/d\tau = df_1(t)/dt. \tag{4}$$

By the indicated correspondence of points in construction of the conformal mappings, Eq. (3) is equivalent to (4) and the conditions

$$\beta(e^{i0}) = e^{i0}, \; \beta(e^{i2\pi}) = e^{i2\pi}. \tag{5}$$

Thus, β is a diffeomorphism of L onto itself, and by the properties of $df_k/d\tau$ the function $\beta'(t)$ satisfies the Hölder condition.

We introduce the functions

$$\Gamma^+(u) = 1 + u d\omega_1/du, \; \Gamma^-(u) = 1 + u d\omega_2/du.$$

Using property (2), we rewrite (4) in the form

$$\Gamma^-(\beta(t)) = \frac{\beta(t)}{t\beta'(t)} \Gamma^+(t). \tag{6}$$

It is well known that the complex potentials of the investigated flows are expressed by the equations [4]

$$w_1 = -i(1 - \gamma) \ln u, \; w_2 = -i(\delta - \gamma) \ln u.$$

Now the velocities squared at one given point of the wave have the form

$$|v_1|^2 = \frac{(1-\gamma)^2}{|\Gamma^+(t)|^2}, \; |v_2|^2 = \frac{(\delta-\gamma)^2}{|\Gamma^-(\beta(t))|^2}. \tag{7}$$

In the interval $[0, 2\pi]$ we define the real function $q(s)$ by the equation

$$\beta(t) = e^{iq(s)}. \tag{8}$$

On the basis of expressions (1), (6)-(8), and the obvious equation

$$t\beta'(t)/\beta(t) = q'(s)$$

we express the shift function $\beta(t)$ in terms of $\Gamma^+(t)$:

$$\beta(t) = \exp\left(\int_{e^{i0}}^{t} \sqrt{\mu + \nu\left(c + \operatorname{Im}\left(i\int_{e^{i0}}^{\tau} \frac{\Gamma^+(\varkappa)}{\varkappa}\, d\varkappa\right) - b\right) |\Gamma^+(\tau)|^2}\, \frac{d\tau}{\tau}\right), \tag{9}$$

where

$$\mu = \frac{m_1(1-\gamma)^2}{(1+m_1)(\delta-\gamma)^2};\ \nu = \frac{2\gamma^2}{\mathrm{Fr}\,(1+m_1)(\delta-\gamma)^2},$$

and the real constant b is determined from the condition that the origin is located at the average level of the liquids:

$$\int_0^{\lambda} Y_l\, dX = 0. \tag{10}$$

We determine the constant c, in turn, from the equation

$$\frac{1}{2\pi i}\int_L \sqrt{\mu + \nu\left(c + \operatorname{Im}\left(i\int_{e^{i0}}^{\tau} \frac{\Gamma^+(\varkappa)}{\varkappa}\, d\varkappa\right) - b\right) |\Gamma^+(\tau)|^2}\, \frac{d\tau}{\tau} = 1, \tag{11}$$

which is deduced from the second condition (5).

We have thus reduced the original problem to the solution of a nonlinear conjugation problem. The latter entails determination of the function $\Gamma^+(u)$, which is analytic inside $|u| < 1$, and the function $\Gamma^-(u)$, which is analytic outside $|u| \leqslant 1$; the limiting values of both functions are Hölder-continuous and satisfy relation (6). Here the function $\beta(t)$ and the constants b and c are evaluated from (9)-(11).

By the hypothesis of a nonvanishing relative velocity at points of the wave we infer from (6), (7), (9), and (11) that the function $\beta(t)$ given by Eq. (9) maps L one-to-one onto itself with preservation of direction and has a nonvanishing derivative β'. Also, since $\Gamma^+(t)$ and $\Gamma^-(t)$ satisfy the Hölder condition, β' also satisfies this condition.

Making use of the fact that $\Gamma^+(t)$ and $\Gamma^-(t)$ are the limiting values of the corresponding analytic functions, we reduce (6) to the integral equation [7]

$$\frac{\Gamma^+(t)}{t} + \frac{1}{2\pi i}\int_L K(t,\tau)\frac{\Gamma^+(\tau)}{\tau}\, d\tau = \frac{1}{t}, \tag{12}$$

in which

$$K(t,\tau) = \beta'(t)/[\beta(\tau) - \beta(t)] - 1/(\tau - t).$$

Defining the function F(t) by the equation

$$\Gamma^+(t) = F(t) + 1,$$

we represent (12) in the operator form

$$F = -\frac{1}{2\pi i}\int_L \frac{t}{\tau} K(t,\tau)(F(\tau)+1)\, d\tau = R(F,\mu,\nu). \tag{13}$$

For any values of the parameters μ and ν, Eq. (13) has the trivial solution $F_0 = 0$ [with $c = (1-\mu)/\nu$, $b = 0$], which corresponds to uniform flow.

We compute the Fréchet derivative $R'(F_0, \mu, \nu; F)$ of the operator $R(F, \mu, \nu)$ at the point F_0 and analyze the following equation linearized at zero:

$$F(t) = (1-\mu)F(t)/2 + (\omega_1(t) - \omega_1(0))\nu/4$$

or

$$F(t) = \frac{\nu}{2(1+\mu)}\int_0^t \frac{1}{u}\frac{1}{2\pi i}\int_L \frac{F(\tau)}{\tau - u}\, d\tau\, du. \tag{14}$$

The spectrum of the operator on the right-hand side of (14) consists of the eigenvalues $2(1 + \mu)/\nu = 1/h$ of multiplicity one.

The corresponding eigenfunctions are t^h, where h is a positive integer. The solution of Eq. (14) has the form αt, where α is a dimensionless amplitude, and the Froude number is

$$\mathrm{Fr} = \gamma^2/(m_1(1-\gamma)^2 + (1+m_1)(\delta-\gamma)^2)h. \tag{15}$$

This equation coincides with the well-known results of the theory of internal waves [8]. In particular, for $m_1 = 0$ and $\delta = 0$ we have $U^2 = g\lambda/2\pi$. We note that a full traversal of L corresponds to one wave. It is sufficient, therefore, to confine the analysis to the value h = 1.

The solution of the nonlinear equation (12) is sought by an iterative procedure. In each iteration we solve the linear equation, in the kernel of which the function $\beta(t)$ is given by Eq. (9), where Γ^+ is the solution of the preceding iteration. For the numerical solution of the linear equation (12) in the n-th iteration it is practical to transform to the equation with a real kernel

$$\Gamma_n^+(s) + \frac{1}{4\pi i}\int_0^{2\pi} K_1(s,\sigma)\,\Gamma_n^+(\sigma)\,d\sigma = \frac{1}{2}(1+q'(s)), \tag{16}$$

in which $K_1(s, \sigma) = q'(s)\cot([q(\sigma) - q(s)])/2 - \cot[(\sigma - s)/2]$ and $q(\sigma)$ is expressed by means of Γ_{n-1}^+ from (8) and (9). Given the condition that Γ_{n-1}^+ satisfies the Hölder condition with exponent α, by the properties of the function $\beta(t)$ $[q(s)]$ the kernel of Eq. (16) has a singularity of lower-than-first order at the point $s = \sigma$:

$$|K_1(s,\sigma)| < M/|s-\sigma|^{1-\alpha},$$

where M is a constant depending on Γ_{n-1}^+. Also, for Γ_{n-1}^+ satisfying the Hölder condition the solution of the equation Γ_n^+ also satisfies this condition.

We limit the discussion to waves for which $1/2 < \alpha \leqslant 1$. We represent the square-summable kernel $K_1(s, \sigma)$ by a Fourier series:

$$K_1(s,\sigma) = \sum_{p,m=0}^{\infty} k_{pm}\eta_p(\sigma)\,\eta_m(s),$$

where

$$\eta_0 = 1/\sqrt{2\pi};\ \eta_{2m-1} = \sin(ms)/\sqrt{\pi};\ \eta_{2m} = \cos(ms)/\sqrt{\pi}.$$

We solve Eq. (16) by the method of moments [9], seeking the solution in the form

$$\Gamma_n^+(s) = 1 + \sum_{m=1}^{n+1}\left(a_m^{(n)} + ib_m^{(n)}\right)e^{ims}.$$

To evaluate the coefficients $a_m^{(n)}$ $b_m^{(n)}$ we obtain a system of linear algebraic equations. We take the solution of Eq. (14) as the initial approximation. We determine the constant c_n from Eq. (11). We test the convergence of the iterative procedure by letting the quantities $c_n - c_{n-1}$ and $\|\Gamma_n^+ - \Gamma_{n-1}^+\|_{L_2}$ tend to zero. The iterations are terminated upon satisfaction of Eq. (11) with error less than or equal to 10^{-6}.

We have carried out calculations for $\delta = 0$ and $m_1 = 0.00129$. In this case the lower liquid is motionless at infinite depth. The results of the calculations show that Eq. (15) in the plane (γ, Fr) determines a branching curve, at each point of which the trivial solution branches into a nontrivial solution corresponding to a particular wave motion (solid curve in Fig. 2). The nontrivial solution is characterized by the ratio H/λ, where H is the wave height. The variation of H/λ as a function of Fr for $\gamma = 0.6$ is represented by the dashed curve in Fig. 3. This dependence agrees quite well with the results of [1], which are represented by the solid curve in Fig. 3. All the solutions obtained here describe waves with a vertical axis of symmetry. The dashed curve in Fig. 2 represents the values of the parameters γ and Fr for which the solution of Eq. (12) corresponds to a wave motion with $H/\lambda = 0.1$. The calculated values of the quantities c_n, $\|\Gamma_n^+ - \Gamma_{n-1}^+\|^2_{L_2}$, and $2\pi - q(2\pi)$, and $q(\pi)$ are given in Table 1 as a function of the iteration number for values Fr = 1.16 and $\gamma = 0.6$. The wave profile obtained for this case is shown in Fig. 4.

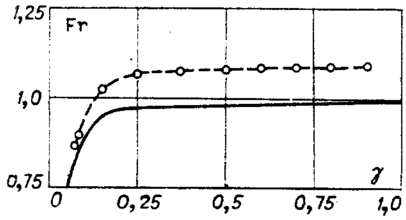

Fig. 2

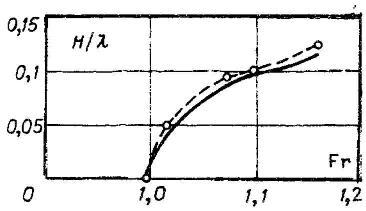

Fig. 2

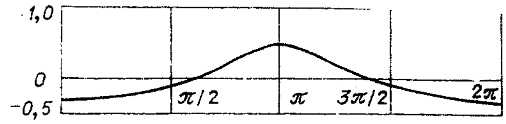

Fig. 3

TABLE 1

Iteration No.	$c_n \cdot 10^{+5}$	$\|\Gamma_n^+ - \Gamma_{n-1}^+\|^2_{L_2} \cdot 10^{+6}$	$2\pi - q(2\pi)$	$q(\pi)$	Iteration No.	$c_n \cdot 10^{+5}$	$\|\Gamma_n^+ - \Gamma_{n-1}^+\|^2_{L_2} \cdot 10^{+6}$	$2\pi - q(2\pi)$	$q(\pi)$
1	57500	80	$38 \cdot 10^{-3}$	3,12	4	58172	49	$96 \cdot 10^{-7}$	3,14
2	58000	90	$87 \cdot 10^{-4}$	3,13	5	58185	37	$72 \cdot 10^{-7}$	3,14
3	58156	68	$61 \cdot 10^{-6}$	3,14	6	58199	29	$79 \cdot 10^{-8}$	3,14

The author is grateful to D. N. Gorelov for stating and discussing the problem, and to V. A. Seleznev for general interest.

LITERATURE CITED

1. N. E. Kochin, "Rigorous determination of steady-state waves of finite amplitude at the surface of separation of two liquids of finite depth," in: Collected Works [in Russian], Vol. 2, Izd. Akad. Nauk SSSR, Moscow (1949).
2. S. R. P. Sinkha, "Rigorous theory of steady-state waves on a free surface and at a surface of separation of two liquids," Dokl. Akad. Nauk SSSR, 168, No. 1 (1966).
3. T. V. Davis, "The theory of symmetrical gravity waves of finite amplitude," Proc. Roy. Soc. London, Ser. A, 208, 475-486 (1951).
4. A. N. Nekrasov, "Rigorous theory of steady-state waves on the surface of a heavy liquid," in: Collected Works [in Russian], Vol. 1, Izd. Akad. Nauk SSSR, Moscow (1962).
5. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of a Complex Variable [in Russian], 4th ed., Nauka, Moscow (1973).
6. G. M. Goluzin, Geometrical Theory of Functions of a Complex Variable [in Russian], 2nd ed., Nauka, Moscow (1966).
7. F. D. Gakhov, Boundary-Value Problems [in Russian], 2nd ed., Fizmatgiz, Moscow (1963).
8. N.E. Kochin, I. A. Kibel', and N. V. Roze, Theoretical Fluid Mechanics [in Russian], Part 1, 6th ed., Fizmatgiz, Moscow (1963).
9. V. V. Ivanov, "Application of the method of moments and a mixed method to the approximate solution of singular integral equations," Dokl. Akad. Nauk SSSR, 114, No. 5 (1957).